Q1

i) FIND MIDPOINT OF AB

$$\left(\frac{-1-7}{2}, \frac{7-5}{2}\right) = (-4, 1)$$

EQUATION OF LINE

$$\overrightarrow{OM} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

$$\overrightarrow{PM} = \begin{pmatrix} -4 \\ 1 \end{pmatrix} - \begin{pmatrix} 5 \\ -6 \end{pmatrix} = \begin{pmatrix} -9 \\ 7 \end{pmatrix}$$

$$\boxed{\underline{r} = \begin{pmatrix} -4 \\ 1 \end{pmatrix} + s\begin{pmatrix} -9 \\ 7 \end{pmatrix}}$$

ii) $\overrightarrow{OB} = \begin{pmatrix} -7 \\ -5 \end{pmatrix}$

PARALLEL = SAME DIRECTION VECTOR

$$\boxed{\underline{r} = \begin{pmatrix} -7 \\ -5 \end{pmatrix} + t\begin{pmatrix} -9 \\ 7 \end{pmatrix}}$$

Q2A

a)

i) $\overrightarrow{OA} = \begin{pmatrix} 6 \\ 1 \\ -3 \end{pmatrix}$

$\overrightarrow{AB} = \begin{pmatrix} -2 \\ -7 \\ 5 \end{pmatrix} - \begin{pmatrix} 6 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -8 \\ -8 \\ 8 \end{pmatrix}$

SIMPLIFY

$$\boxed{\underline{r} = \begin{pmatrix} 6 \\ 1 \\ -3 \end{pmatrix} + s\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}}$$

ii) $\overrightarrow{OB} = \begin{pmatrix} -2 \\ -7 \\ 5 \end{pmatrix}$ $\quad \overrightarrow{OC} = \begin{pmatrix} 3 \\ 12 \\ -9 \end{pmatrix}$

SIMPLIFY

$$\boxed{\underline{r} = \begin{pmatrix} -2 \\ -7 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}}$$

SIMPLIFYING NOT ESSENTIAL BUT GOOD PRACTICE

Q2B

b) FIND VALUE OF S AND t WHICH WORKS FOR ALL PARTS OF VECTOR

$$\underline{r}_1 = \begin{pmatrix} 6-s \\ 1-s \\ -3+s \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix} \qquad \begin{matrix} s=5 \\ s=-4 \\ s=-1 \end{matrix}$$

$$\boxed{\text{So } (1,5,-4) \text{ DOES NOT LIE ON } \underline{r}_1}$$

$$\underline{r}_2 = \begin{pmatrix} -2+t \\ -7+4t \\ 5-3t \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix} \qquad \begin{matrix} t=3 \\ t=3 \\ t=3 \end{matrix}$$

$$\boxed{\text{So } (1,5,-4) \text{ DOES LIE ON } \underline{r}_2}$$

Q3

The coordinates of three points are $A(-2,3,-5)$, $B(2,-5,-9)$ and $C(-10,-1,-1)$.

The point M is the midpoint of AB, and the point N lies on BC.

Given that $|\overrightarrow{BN}| = 3|\overrightarrow{NC}|$, find the equation of the line through points M and N in vector form.

[6]

MIDPOINT AB = M

$$M = \left(\frac{-2+2}{2}, \frac{3-5}{2}, \frac{-5-9}{2}\right) = (0,-1,-7)$$

$\overrightarrow{BC}$ $\quad \overrightarrow{OC} - \overrightarrow{OB}$

$$\begin{pmatrix} -10 \\ -1 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ -5 \\ -9 \end{pmatrix} = \begin{pmatrix} -12 \\ 4 \\ 8 \end{pmatrix}$$

FIND $\overrightarrow{ON}$ USING $|\overrightarrow{BN}| = 3|\overrightarrow{NC}|$

$$\overrightarrow{ON} = \overrightarrow{OB} + \tfrac{3}{4}\overrightarrow{BC}$$

$$\begin{pmatrix} 2 \\ -5 \\ -9 \end{pmatrix} + \tfrac{3}{4}\begin{pmatrix} -12 \\ 4 \\ 8 \end{pmatrix} = \begin{pmatrix} -7 \\ -2 \\ -3 \end{pmatrix}$$

EQUATION THROUGH M AND N

$$\overrightarrow{OM} = \begin{pmatrix} 0 \\ -1 \\ -7 \end{pmatrix}$$

$$\overrightarrow{MN} = \begin{pmatrix} -7 \\ -2 \\ -3 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ -7 \end{pmatrix} = \begin{pmatrix} -7 \\ -1 \\ 4 \end{pmatrix}$$

$$\boxed{\underline{r} = \begin{pmatrix} 0 \\ -1 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ -1 \\ 4 \end{pmatrix}}$$

OR EQUIVALENT

Q4

$$\overrightarrow{CA} = \begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \\ -4 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix}$$

$$\overrightarrow{CB} = \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

SCALAR PRODUCT

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = -6 - 5 + 4 = -7$$

MAGNITUDE

$$|\overrightarrow{CA}| = \sqrt{(-2)^2 + 5^2 + 4^2} = \sqrt{45}$$

$$|\overrightarrow{CB}| = \sqrt{3^2 + (-1)^2 + 1^2} = \sqrt{11}$$

$$\theta = \cos^{-1}\left(\frac{-7}{\sqrt{45}\sqrt{11}}\right) = 108.3382\ldots$$

$$\boxed{108.3^\circ \text{ (1dp)}}$$

Q5

The vertices of triangle ABC are the points with coordinates $A(-2,5,4)$, $B(3,1,0)$ and $C(-1,-3,-1)$.

Use a vector method to prove that ABC is a right-angled triangle.

[4]

VARIOUS VALID VECTOR METHODS

e.g.

SCALAR $a \cdot b = 0$ PERPENDICULAR

OR

MAGNITUDE OF VECTORS + SHOW PYTHAGORAS WORKS

$\overrightarrow{AB} \cdot \overrightarrow{AC}$

$$\begin{pmatrix} 5 \\ -4 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -8 \\ -5 \end{pmatrix} = 5 + 32 + 20 = 57$$

$\overrightarrow{BA} \cdot \overrightarrow{BC}$

$$\begin{pmatrix} -5 \\ 4 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ -4 \\ -1 \end{pmatrix} = 20 - 16 - 4 = 0$$

$\overrightarrow{CA} \cdot \overrightarrow{CB}$

$$\begin{pmatrix} -1 \\ 8 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} = -4 + 32 + 5 = 33$$

$\boxed{\overrightarrow{BA} \cdot \overrightarrow{BC} = 0 \text{ SO TRIANGLE ABC IS RIGHT ANGLED}}$

Q6

i) $\begin{pmatrix} 3+s \\ -5+3s \\ 1-2s \end{pmatrix} = \begin{pmatrix} 1+t \\ 2-t \\ 2-t \end{pmatrix}$ RHS EQUAL

$\therefore$ LHS EQUAL

SOLVE LHS

$$-5+3s = 1-2s \qquad s = \frac{6}{5}$$

$$5s = 6$$

SUB AND SOLVE FOR t

$$-5+3\left(\frac{6}{5}\right) = 2-t \qquad t = 7 - \frac{18}{5} = \frac{17}{5}$$

CHECK IN 1ST EQUATION

$$3+\frac{6}{5} \neq 1+\frac{17}{5} \quad \times \quad \frac{21}{5} \neq \frac{22}{5}$$

LINES DO NOT INTERSECT

CHECK IF VECTORS ARE PARALLEL

$$\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \mu \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$\boxed{\text{DIRECTION VECTORS ARE NOT MULTIPLES OF EACH OTHER SO LINES ARE SKEW}}$

ii)
$$\begin{array}{c} ① \\ ② \\ ③ \end{array}\begin{pmatrix} 1+2s \\ -2+6s \\ 3-2s \end{pmatrix} = \begin{pmatrix} 4-5t \\ 7-15t \\ 5t \end{pmatrix}$$

SUB ③ INTO ①

$$1+2s = 4-(3-2s)$$

$$1+2s = 1+2s \qquad \text{S CAN BE ANY VALUE}$$

SUB $s=0$ INTO ③

$$3 = 5t \qquad t = \frac{3}{5}$$

CHECK IN ②

$$-2+6(0) = 7-15\left(\frac{3}{5}\right)$$

$$-2 = -2 \checkmark\checkmark \qquad \text{LINES INTERSECT}$$

$s = \mathbb{R}$ LINES MUST BE THE SAME

$$\begin{pmatrix} 2 \\ 6 \\ -2 \end{pmatrix} = \mu\begin{pmatrix} -5 \\ -15 \\ 5 \end{pmatrix}$$

$\boxed{\mu = -\frac{5}{2} \text{ LINES ARE PARALLEL AND INTERSECT SO ARE THE SAME LINE}}$

iii) SIMPLIFY DIRECTION VECTORS

USE μ AND λ TO HIGHLIGHT CHANGED EQUATIONS

$$\underline{r} = -i-4j-4k+\mu(i-j-2k)$$

$$\underline{r} = -4i-j-2k+\lambda(i-j+2k)$$

$$\begin{array}{c} ① \\ ② \\ ③ \end{array}\begin{pmatrix} -1+\mu \\ -4-\mu \\ -4-2\mu \end{pmatrix} = \begin{pmatrix} -4+\lambda \\ -1-\lambda \\ -2+2\lambda \end{pmatrix}$$

SOLVE SIMULTANEOUSLY $2\times① + ③$

$$-2+2\mu = -8+2\lambda$$

$$(+)\ -4-2\mu = -2+2\lambda \qquad \lambda = 1$$

$$-6 = -10+4\lambda$$

SUB INTO ① $-1+\mu = -4+1 \qquad \mu = -2$

CHECK IN ② $-4-(-2) = -1-1 \qquad -2 = -2 \checkmark$

SUB EITHER μ OR λ INTO SIMPLIFIED EQUATIONS FOR POINT OF INTERSECTION

USING UNSIMPLIFIED EQUATIONS TO SOLVE FOR S AND t WILL STILL GIVE SAME FINAL POINT OF INTERSECTION

$$\begin{pmatrix} -1+(-2) \\ -4-(-2) \\ -4-2(-2) \end{pmatrix} \text{ OR } \begin{pmatrix} -4+1 \\ -1-1 \\ -2+2 \end{pmatrix} = \boxed{\text{LINES INTERSECT AT } \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix} \text{ OR } (-3,-2,0)}$$

Q7

Find the coordinates of the point on the line $\mathbf{r} = 2\mathbf{i} - 12\mathbf{j} + 3\mathbf{k} + s(\mathbf{i} - 6\mathbf{j} + 4\mathbf{k})$ that is closest to the point $P(2, 3, -1)$, and hence determine the minimum distance from point P to the line.

[6]

PERPENDICULAR = SHORTEST DISTANCE

SCALAR PRODUCT = 0

FOR SOME POINT ON THE LINE N

$$\overrightarrow{ON} = \begin{pmatrix} 2 \\ -12 \\ 3 \end{pmatrix} + s\begin{pmatrix} 1 \\ -6 \\ 4 \end{pmatrix}$$ FOR SOME VALUE OF S

$$\overrightarrow{OP} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$$\overrightarrow{NP} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 2+s \\ -12-6s \\ 3+4s \end{pmatrix} = \begin{pmatrix} -s \\ 15+6s \\ -4-4s \end{pmatrix}$$

$$\overrightarrow{NP} \cdot \text{DIRECTION} \begin{pmatrix} 1 \\ -6 \\ 4 \end{pmatrix} = 0$$

$$\begin{pmatrix} -s \\ 15+6s \\ -4-4s \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -6 \\ 4 \end{pmatrix} = -s - 90 - 36s - 16 - 16s$$

$$-106 - 53s = 0$$

$$s = -\frac{106}{53} = -2$$

$\overrightarrow{ON}$ WHEN $s = -2$

$$\begin{pmatrix} 2-2 \\ -12+12 \\ 3-8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix} \Rightarrow \boxed{\text{CLOSEST POINT } (0, 0, -5)}$$

SHORTEST DISTANCE = $|\overrightarrow{NP}|$

$$|\overrightarrow{NP}| = \begin{pmatrix} -(-2) \\ 15+6(-2) \\ -4-4(-2) \end{pmatrix} = \sqrt{2^2 + 3^2 + 4^2}$$

$$\boxed{\sqrt{29} \text{ UNITS}}$$

Q8

FIND EQUATION $\overrightarrow{BD}$

$$\underline{r} = \begin{pmatrix} 3 \\ 23 \\ 3 \end{pmatrix} + s\begin{pmatrix} 27 \\ -9 \\ 9 \end{pmatrix} \Rightarrow \underline{r} = \begin{pmatrix} 3 \\ 23 \\ 3 \end{pmatrix} + s\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

SIMPLIFY

ALSO EQUATION FOR POINT N FOR SOME VALUE OF S $\Rightarrow$ CLOSEST POINT TO A $\overrightarrow{AN} \cdot \text{DIRECT} \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = 0$

$$\overrightarrow{AN} = \begin{pmatrix} 3+3s \\ 23-s \\ 3+s \end{pmatrix} - \begin{pmatrix} 19 \\ 47 \\ 23 \end{pmatrix} = \begin{pmatrix} -16+3s \\ -24-s \\ -20+s \end{pmatrix}$$

SCALAR PRODUCT

$$\begin{pmatrix} -16+3s \\ -24-s \\ -20+s \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = 0$$

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$$-48 + 9s + 24 + s - 20 + s = 0$$

$$-44 + 11s = 0$$

$$s = 4$$

$$\overrightarrow{ON} = \begin{pmatrix} 3+3s \\ 23-s \\ 3+s \end{pmatrix} \Rightarrow \begin{pmatrix} 15 \\ 19 \\ 7 \end{pmatrix}$$ (ON LIES ON $\overrightarrow{BD}$ AND $\overrightarrow{CS}$)

$ON = \begin{pmatrix} 15 \\ 19 \\ 7 \end{pmatrix}$ FIND EQUATION $\overrightarrow{CS}$ (USE $\overrightarrow{ON} - \overrightarrow{OC}$ FOR DIRECTION)

$$\overrightarrow{CS} = \underline{r} = \begin{pmatrix} 26 \\ 63 \\ -4 \end{pmatrix} + t\begin{pmatrix} -11 \\ -44 \\ 11 \end{pmatrix} \Rightarrow \begin{pmatrix} 26 \\ 63 \\ -4 \end{pmatrix} + t\begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix}$$

SIMPLIFY

$$|\overrightarrow{CS}| = 45\sqrt{2}$$

FIND MAGNITUDE OF DIRECTION OF $\overrightarrow{CS}$

$$\left|\begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix}\right| = \sqrt{(-1)^2 + (-4)^2 + 1^2} = \sqrt{18} = 3\sqrt{2}$$

$$3\sqrt{2}\,t = 45\sqrt{2}$$

DISTANCE DOES NOT INDICATE DIRECTION

TWO POSSIBILITIES $t = 15$ OR $t = -15$

$t = 15$ $\qquad$ $t = -15$

$$\begin{pmatrix} 26 - (15) \\ 63 - 4(15) \\ -4 + (15) \end{pmatrix} = \begin{pmatrix} 11 \\ 3 \\ 11 \end{pmatrix} \text{ OR } \begin{pmatrix} 26 - (-15) \\ 63 - 4(-15) \\ -4 + (-15) \end{pmatrix} = \begin{pmatrix} 41 \\ 123 \\ -19 \end{pmatrix}$$

$$\boxed{S = (11, 3, 11)}$$